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Barut-Girardello Coherent States for the Parabolic Cylinder Functions

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Abstract The Barut-Girardello coherent states for the parabolic cylinder functions are constructed. It is shown that the resolution of unity condition is satisfied for the coherent states. In the Hilbert space spanned by the parabolic cylinder eigenstates, the appropriate measure is also obtained.

Keywords Coherent States · Special Functions · Quantum Mechanics

1 Introduction

In 1926, Schrödinger introduced coherent states in an attempt to find quantum-mechanical states which provide a close connection between quantum and classical formulations of a given physical system [1]. These states are corresponded to the Heisenberg-Weyl group whose Lie algebra is given in terms of the harmonic oscillator creation and annihilation operators. However, between 1926 and 1963, activities in the field of coherent states were not remarkable. In fact, thirty-five years after Schrödinger's pioneering idea, Glauber and Sudarshan [2–5] made the first application of the coherent states. Glauber constructed the eigenstates of the annihilation operator of the harmonic oscillator in order to study the electromagnetic correlation functions in the context of the quantum optics. The original coherent states introduced by Schrödinger were extended to a large number of Lie groups with square integrable representations [6, 7]. These extensions have many applications in all branches of quantum physics. Indeed, Klauder [8, 9] developed a set of continuous states in which

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the basic theory of coherent states for arbitrary Lie groups were considered. Then, the complete construction of coherent states of Lie groups with various properties was performed by Perelomov and Gilmore [10–12]. The basic idea of this development was to relate the coherent states with the dynamical symmetry group for each physical system.

There are three definitions of coherent states. The first is to define the coherent states as eigenstates of an annihilation operator of a given Lie group with complex eigenvalues. This definition is called the Barut-Girardello coherent states [13]. In 1999, Gazeau and Klauder [14] suggested new coherent states which can be considered as a generalization of the Barut-Girardello coherent states. The second definition, often called Klauder-Perelomov coherent states [8–10], is to introduce the coherent states by action of an unitary z-displacement ($z \in \mathcal{C}$) operator on the ground state of the system. The third definition is to consider the coherent states as quantum states which give the minimum Heisenberg uncertainty relations for the Hermitian generators of a Lie group [15–18]. The last definition was first introduced by Schrödinger for the harmonic oscillator. In fact, the mentioned definitions overlap only for the special case of Heisenberg-Weyl group which is the dynamical symmetry group of the quantum harmonic oscillator.

Coherent states for systems other than the harmonic oscillator have attracted much attention for several years [19–31]. The parabolic cylinder functions have important applications in theoretical physics. In fact, the wave functions of many quantum systems are represented by the parabolic cylinder functions [32–35]. In this paper we construct the Barut-Girardello coherent states for the parabolic cylinder functions. We realize the resolution of unity condition for the coherent states. Then the overlapping of two different coherent states is derived. In the Hilbert space spanned by the parabolic cylinder eigenstates, the appropriate measure is obtained.

2 Barut-Girardello Coherent States for the Parabolic Cylinder Functions

In this section we construct the Barut-Girardello Coherent States for the parabolic cylinder functions. One may show the normalized parabolic cylinder wave functions by

$$\psi_n(x) = C_n D_n(x) \tag{1}$$

where $D_n(x)$ is the parabolic cylinder function and C_n is the normalization factor. Using the following integral relation [36]

$$\int_{-\infty}^{+\infty} dx\, |D_n(x)|^2 = \sqrt{2\pi}\, n!, \tag{2}$$

we find

$$C_n = \frac{1}{\sqrt{\sqrt{2\pi}\, n!}}. \tag{3}$$

On the other hand, the parabolic cylinder functions satisfy the following recursion relations [36]

$$D_{n+1}(x) - x D_n(x) + n D_{n-1}(x) = 0, \tag{4}$$

$$\frac{d}{dx} D_n(x) + \frac{1}{2} x D_n(x) - n D_{n-1}(x) = 0, \tag{5}$$

$$\frac{d}{dx} D_n(x) - \frac{1}{2} x D_n(x) + D_{n+1}(x) = 0, \tag{6}$$

which, in turn, lead to

$$\left(\frac{1}{2}x + \frac{d}{dx}\right)\psi_n(x) = \sqrt{n}\,\psi_{n-1}(x), \tag{7}$$

$$\left(-\frac{1}{2}x + \frac{d}{dx}\right)\psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x). \tag{8}$$

Thus, the raising and lowering operators can be defined as

$$A^- = \frac{1}{2}x + \frac{d}{dx}, \tag{9}$$

$$A^+ = -\frac{1}{2}x + \frac{d}{dx}. \tag{10}$$

So, it is evident that

$$A^-\psi_n(x) = \sqrt{n}\,\psi_{n-1}(x), \tag{11}$$

$$A^+\psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x). \tag{12}$$

Meanwhile, we have

$$(A^-)^n\,\psi_n(x) = \sqrt{n!}\,\psi_0(x), \tag{13}$$

$$(A^+)^n\,\psi_0(x) = \sqrt{n!}\,\psi_n(x). \tag{14}$$

The Barut-Girardello coherent states are defined as the eigenstates of the lowering operator:

$$A^-|z\rangle = z|z\rangle \tag{15}$$

where z is an arbitrary complex variable. Using (11), we can obtain the coherent states $|z\rangle$ as a linear combination of the eigenstates of the parabolic cylinder functions:

$$|z\rangle = M(z)^{-1}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}z^n\,|\psi_n(x)\rangle \tag{16}$$

where $M(z)$ is the normalization factor. Some calculations shows that

$$M(z)^2 = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!} = e^{|z|^2}. \tag{17}$$

Therefore, the Barut-Girardello coherent states for the parabolic cylinder functions take the following form

$$|z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}z^n\,|\psi_n(x)\rangle. \tag{18}$$

Applying the relations (1) and (3), we have

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{2\pi}\,n!}}D_n(x). \tag{19}$$

On the other hand, regarding the following differential representation of the parabolic cylinder function [36]

$$D_n(x) = (-1)^n e^{-x^2/4} \frac{d^n}{dx^n}(e^{-x^2/2}), \tag{20}$$

one may obtain

$$\psi_n(x) = \frac{e^{x^2/4}}{\sqrt{n!}} D_n(x)\psi_0(x). \tag{21}$$

Now, the Barut-Girardello coherent states become

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{x^2/4} D_n(x)\, |\psi_0(x)\rangle. \tag{22}$$

Furthermore, using the following series [37]

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} D_{n+\nu}(x) = e^{z(2x-z)/4} D_\nu(x-z), \tag{23}$$

and after some manipulations we obtain the Barut-Girardello coherent states for the parabolic cylinder functions as

$$|z\rangle = e^{-\frac{1}{2}(z^2-2zx+|z|^2)}\, |\psi_0(x)\rangle. \tag{24}$$

The overlapping of two different coherent states for two arbitrary complex variables z_1 and z_2 is calculated as follows:

$$\langle z_1|z_2\rangle = e^{-\frac{1}{2}(|z_1|^2+|z_2|^2-2\bar{z}_1 z_2)}. \tag{25}$$

Now, we should find the appropriate measure $d\mu(z)$ such that the resolution of unity condition is realized for the coherent states $|z\rangle$:

$$\int d\mu(z)\, |z\rangle\langle z| = 1. \tag{26}$$

It is better to represent the complex variable z in the polar coordinates as $z = re^{i\phi}$ and to consider the measure

$$d\mu(z) = \mu(r) r dr d\phi. \tag{27}$$

If $|f\rangle$ and $|g\rangle$ are two arbitrary state vector in the Hilbert space spanned by the parabolic cylinder eigenstates, then

$$\langle f|g\rangle = \int d\mu(z)\langle f|z\rangle\langle z|g\rangle. \tag{28}$$

Using the relation (18), the above equation becomes (after some calculations)

$$\langle f|g\rangle = 2\pi \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \int_0^\infty dr\, \mu(r) r^{2n+1} e^{-r^2} \right\} \langle f|\psi_n\rangle\langle\psi_n|g\rangle. \tag{29}$$

So, we must have

$$2\pi \int_0^\infty dr\,\mu(r) r^{2n+1} e^{-r^2} = n! \tag{30}$$

which leads to

$$\mu(r) = \frac{1}{\pi}. \tag{31}$$

Finally, we obtain the appropriate measure as

$$d\mu(z) = \frac{1}{\pi} d^2 z. \tag{32}$$

3 Conclusion

Coherent states are used in all branches of quantum physics, for example, quantum optics, nuclear, atomic, and solid-state physics, quantum electrodynamics, quantization problems and path integrals. The basis vectors of the Hilbert space corresponding to the coherent states, are some special functions of mathematical physics. The parabolic cylinder functions have important applications in quantum physics. For example, the wave functions of many quantum systems are expressed in terms of the parabolic cylinder functions. In this paper, the parabolic cylinder functions are considered as the basis vector of the Hilbert space. Raising and lowering operators are constructed. Then, the Barut-Girardello coherent states for the parabolic cylinder functions are calculated. The resolution of unity and the overlapping properties are also investigated. Finally in the Hilbert space spanned by the parabolic cylinder eigenstates, the appropriate measure is obtained. It is obvious that other special functions can be studied by the same method used here.

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